

On well-posedness for power type coupled nonlinear Shrödinger equations in \mathbb{R}

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ABSTRACT

We consider the initial value problem for the coupled Shrödinger equations of one dimensional space:

$$\begin{cases} i\partial_t u_j + \Delta u_j = \gamma \sum_{k=1}^m |u_k|^p |u_j|^{p-2} u_j, t \in \mathbb{R}, \\ u_j(0, x) = \psi_j(x), x \in \mathbb{R}, \end{cases}$$

where $m \geq 2$ is an integer and $\gamma \in \mathbb{R}^*$. At first, assuming $s > \frac{1}{2}$ and $p \geq 2$ is an even integer, we establish local existence and uniqueness of solution for this Cauchy problem with initial data $\psi = (\psi_j)_{1 \leq j \leq m} \in (H^s)^m$. The second part is devoted to global well-posedness of the problem with L^2 -initial data when $p \in [2, 3[$. Let us remark that assumption $p \geq 2$ seems to be technical and yields to the restriction of studying the one dimensional problem.

KEYWORDS

Nonlinear Schrödinger system, well-posedness, power type, mass conservation, iteration principle.

1. Introduction

In this paper, we consider the m -component coupled nonlinear Shrödinger equations with power type nonlinearities which is denoted $(CNLS)_p$:

$$\begin{cases} i\partial_t u_j + \Delta u_j = \gamma \sum_{k=1}^m |u_k|^p |u_j|^{p-2} u_j, t \in \mathbb{R}, \\ u_j(0, x) = \psi_j(x), x \in \mathbb{R}. \end{cases} \quad (1)$$

Here $m \geq 2$ is an integer, $u_j(t, x)$, $\psi_j(x)$ ($1 \leq j \leq m$) are complex valued functions and $\gamma \in \mathbb{R}^*$. Let us recall that single equation is called the nonlinear Shrödinger equation

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which is denoted $(NLS)_p$:

$$\begin{cases} i\partial_t u + \Delta u = \gamma|u|^{p-1}u, t \in \mathbb{R}, \\ u(0, x) = \psi(x), x \in \mathbb{R}, \end{cases} \quad (2)$$

where $u(t, x)$ and $\psi(x)$ are complex valued functions and $\gamma \in \mathbb{R}^*$. This equation arises in many physical problems. In fact $(NLS)_p$ comes from the theory of quantum mechanics. It is also one of the most universal model which describes the evolution of a wave packet in weakly nonlinear and dispersive media. In particular, when $p = 3$, the equation $(NLS)_3$ occurs to model the propagation of waves in optical fibres. When the dimension of the space exceeds two, the focusing of laser beams and the Bose-Einstein condensation phenomenon are modeled by the $(NLS)_p$. The problem of well-posedness of the one component equation $(NLS)_p$ has been studied and widely investigated, for example we can review [3, 4, 7, 14, 15] and references therein.

Concerning $(CNLS)_p$, this kind of m -component coupled nonlinear Schrödinger equations models physical system in which the field has more than one component, for example in optical fibres and wave guides, the propagating electric field has two components that are transverse to the direction of propagation. The $(CNLS)_p$ system arises in the Hartree-Fock theory for a two components Bose-Einstein condensate. For the derivation and more applications of the system $(CNLS)_p$, we refer to [1, 5, 6, 18, 19]. Before we proceed to the discussion, it is useful to look at the most vital symmetry of the equation $(CNLS)_p$ which is scaling. For every $\lambda > 0$ and $l \in \mathbb{R}$, we introduce $u_{j,\lambda}(t, x) = \lambda^l u_j(\lambda^2 t, \lambda x)$, one can observe that:

$$\partial_t u_{j,\lambda}(t, x) = \lambda^{l+2} (\partial_t u_j)(\lambda^2 t, \lambda x); \quad \Delta u_{j,\lambda}(t, x) = \lambda^{l+2} (\Delta u_j)(\lambda^2 t, \lambda x).$$

Then, it follows that:

$$i\partial_t u_{j,\lambda} + \Delta u_{j,\lambda} = \gamma \frac{\lambda^{l+2}}{\lambda^{l(2p-1)}} \sum_{k=1}^m |u_{k,\lambda}|^p |u_{j,\lambda}|^{p-2} u_{j,\lambda}.$$

So, the function $u_\lambda = (u_{j,\lambda})_{1 \leq j \leq m}$ is a symmetrical solution for the Cauchy problem $(CNLS)_p$ if the power of λ equals to zero, that is $l = \frac{1}{p-1}$. Observe that

$$\|u_{j,\lambda}(t, \cdot)\|_{L^2} = \lambda^{\frac{1}{p-1} - \frac{1}{2}} \left(\int_{\mathbb{R}} |u_j(\lambda^2 t, y)|^2 dy \right)^{\frac{1}{2}} = \lambda^{\frac{1}{p-1} - \frac{1}{2}} \|u_j(\lambda^2 t, \cdot)\|_{L^2}.$$

Then, the L^2 norm is unaffected by scaling when $p = 3$, this exponent is called the mass critical exponent. For smaller p , that is $p \in]1, 3[$ which is called the mass subcritical exponent, contracting time reduces the size of the L^2 norm. This is the effect that will be exploited to build up solutions by introducing Duhamel's formula and an iteration principle. Unlike in the subcritical regime, contracting time for larger p , that is $p \in]3, +\infty[$, increases solution norm. This makes more difficulties to work within that region, we are in the mass supercritical case. It is the aim of this paper to establish the well-posedness results for the $(CNLS)_p$, by looking in a parallel way to what we have for the one component equation $(NLS)_p$. Thus, to establish local existence for $(CNLS)_p$ with initial data in $(L^2)^m$, we use contraction mapping techniques based on Strichartz estimates, see [2, 3, 15, 16]. Intensive work has been done in the last few

years about the coupled Schrödinger equations, see [10–13, 17] and references therein. But, the most of these works treat the problem $(CNLS)_p$ with initial data in the energy space $(H^1)^m$. Despite the partial progress made so far, many difficult questions remain open and little is known about the Cauchy problem $(CNLS)_p$ for initial data in $(L^2)^m$. Actually, the problem with initial data in $(L^2)^m$ is still open for higher dimension.

The manuscript is organized as follows. Section two summarizes the main results established in this paper. Section three presents the tools needed for the existence results proved here. Section four is devoted to proving well-posedness of the problem $(CNLS)_p$ with regular initial data. Finally, in section five we give a proof of the global well-posedness of our system for much rougher initial data, that is $\psi = (\psi_j)_{1 \leq j \leq m} \in (L^2)^m$.

2. Main results

We first introduce some notations. In this work, we will consider the Lebesgue spaces $L^p(\mathbb{R})$ equipped with norms

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}; \quad \|f\|_{\infty} := \sup_{x \in \mathbb{R}} \text{ess} |f(x)|.$$

Taking s in \mathbb{R} , we consider the fractional Sobolev space $H^s(\mathbb{R})$ which is the set of functions f in $L^2(\mathbb{R})$ for which

$$\|f\|_{H^s} := \|(1 + |\xi|^2)^{\frac{s}{2}} \mathfrak{F}[f](\xi)\|_2$$

is finite, here $\mathfrak{F}[f]$ is the Fourier transform of f which is defined as follows:

$$\mathfrak{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \exp(-ix\xi) dx.$$

The Fourier transform is defined on the class of the Schwartz function and then extended to tempered distribution. Its inverse operator is defined as the following:

$$\mathfrak{F}^{-1}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \exp(ix\xi) dx.$$

We need to define some Böchner spaces on I an interval of time which can be closed if necessary:

$$L^q(I, L^r(\mathbb{R})); L^q(I, H^s(\mathbb{R})); C(I, L^r(\mathbb{R})) \text{ and } C(I, H^s(\mathbb{R})).$$

For the borel-mesurable function $u : (t, x) \in I \times \mathbb{R} \rightarrow \mathbb{C}$, we define the norms of the spaces listed above

$$\|u\|_{L^q L^r(I \times \mathbb{R})} := \left(\int_I \|u(t, \cdot)\|_r^q dt \right)^{\frac{1}{q}}; \quad \|u\|_{L^q H^s(I \times \mathbb{R})} := \left(\int_I \|u(t, \cdot)\|_{H^s}^q dt \right)^{\frac{1}{q}}, \quad 1 \leq q, r < \infty,$$

$$\|u\|_{CL^r(I \times \mathbb{R})} := \sup_{t \in I} \|u(t, \cdot)\|_r ; \|u\|_{CH^s(I \times \mathbb{R})} := \sup_{t \in I} \|u(t, \cdot)\|_{H^s}, 1 \leq r \leq \infty.$$

Also, we need to define the space $\mathbf{S}(I \times \mathbb{R})$ which is governed by all Strichartz norms $\|\cdot\|_{L^q L^r(I \times \mathbb{R})}$ where (q, r) satisfies the admissibility conditions:

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}; 2 \leq q, r \leq \infty.$$

So, for any function $f \in \mathbf{S}(I \times \mathbb{R})$, let

$$\|f\|_{\mathbf{S}(I \times \mathbb{R})} := \sup_{(q,r) \text{ admissible}} \|f\|_{L^q L^r(I \times \mathbb{R})}.$$

Then, the dual space of $\mathbf{S}(I \times \mathbb{R})$, which will be called $\mathbf{N}(I \times \mathbb{R})$, is equipped with the following standard norm:

$$\|g\|_{\mathbf{N}(I \times \mathbb{R})} = \sup_{f \in \mathbf{S}(I \times \mathbb{R}), \|f\|_{\mathbf{S}(I \times \mathbb{R})} \leq 1} \int_{I \times \mathbb{R}} g(t, x) \cdot f(t, x) dt dx$$

If X is an abstract space, then X^m will denote the product space $X \times X \times \dots \times X$ with m components. Moreover, if $u = (u_j)_{1 \leq j \leq m} \in X^m$ and $\|\cdot\|_X$ is the norm of X , then $\|u\|_{X^m} := \sup_{1 \leq j \leq m} \|u_j\|_X$ is the norm of X^m . In the remainder of this paper, we

will note $u = (u_j)$ without any precision for the range of j .

Once we have introduced notations that we need. We say that an initial value problem for partial differential coupled Schrödinger equations is local well-posed in some given space X if

- there exists a time interval $[-T, T]$, in which the problem has a solution in X ,
- the solution is unique in X ,
- the solution depends continuously on the initial data.

We prove at first the following result which is crucial to produce some bounds for the nonlinear term.

Lemma 1. For any $a, b, \alpha, \beta \in \mathbb{C}$ and $p \geq 2$, we have

$$\left| |a|^p |b|^{p-2} b - |\alpha|^p |\beta|^{p-2} \beta \right| \leq 2p \{ |a|^{2(p-1)} + |b|^{2(p-1)} + |\alpha|^{2(p-1)} + |\beta|^{2(p-1)} \} (|a - \alpha| + |b - \beta|).$$

Now, we need to transform inequality obtained in the previous Lemma in to inequality with the $H^s(\mathbb{R})$ norm. Let us remark that $\| |f| \|_{H^s}$ and $\| f \|_{H^s}$ may be quite different. To avoid dealing with absolute value, we make use of the expansion $|f|^2 = f \cdot \bar{f}$ and the fact that $\| f \|_{H^s} = \| \bar{f} \|_{H^s}$. Therefore, we are most interested in the case where p is an even integer. So, the nonlinearity can be expressed as:

$$|u_k|^p |u_j|^{p-2} u_j = u_k^{\frac{p}{2}} u_j^{\frac{p}{2}} \bar{u}_k^{\frac{p}{2}} \bar{u}_j^{\frac{p-2}{2}}.$$

Lemma 2. Let $s > \frac{1}{2}$ and $q \in \mathbb{N}^*$. Then, the following inequality holds for any

$u, v \in H^s(\mathbb{R})$:

$$\|u^q - v^q\|_{H^s} \leq C(s, q) \|u - v\|_{H^s} \sum_{k=0}^{q-1} \|u\|_{H^s}^{q-1-k} \|v\|_{H^s}^k,$$

where $C(s, q) > 0$ is a real constant depending on s and q .

Lemma 3. *Let $s > \frac{1}{2}$ be a real number and $p \geq 2$ be an even integer. Then, for all $u, v, f, g \in H^s(\mathbb{R})$ we have*

$$\| |u|^p |v|^{p-2} v - |f|^p |g|^{p-2} g \|_{H^s} \leq C(s, p) \left\{ \sum_{h \in \{u, v, f, g\}} \|h\|_{H^s}^{2(p-1)} \right\} (\|u - f\|_{H^s} + \|v - g\|_{H^s}).$$

Here $C(s, p) > 0$ is a real constant.

Based on these Lemma and a fixed point method, we prove the local well-posedness of the problem 1 in $H^s(\mathbb{R})^m$ for some range of the exponent p .

Theorem 1. *Let $s > \frac{1}{2}$ be a real number and $p \geq 2$ be an even integer. Assume that $\psi = (\psi_j)$ is in $H^s(\mathbb{R})^m$, then*

- for all $R > 0$, there exist $T = T(R, s, p, \gamma, m) > 0$ such that, if $\|\psi\|_{H^s(\mathbb{R})^m} \leq R$, then there is $u = (u_j) \in C([-T, T], H^s(\mathbb{R})^m)$ that solves the problem $(CNLS)_p$ on $[-T, T]$,
- u is the unique function in $C([-T, T], H^s(\mathbb{R})^m)$ that solves $(CNLS)_p$,
- the solution depends continuously on the initial data.

Now we state that global well-posedness holds in the Strichartz space $\mathbf{S}(\mathbb{R} \times \mathbb{R})^m$ with any data in $L^2(\mathbb{R})^m$ and a subcritical exponent $p \in [2, 3[$. Notice that the L^2 -theory does not see the difference between the defocusing case $\gamma > 0$ and the focusing case $\gamma < 0$.

Theorem 2. *Let p be a real number such that $p \in [2, 3[$ and assume that $\psi = (\psi_j) \in L^2(\mathbb{R})^m$. Then we have*

- for all $R > 0$, there exist $T = T(R, p, \gamma, m) > 0$ such that, if $\|\psi\|_{L^2(\mathbb{R})^m} \leq R$, then there is $u = (u_j) \in C([-T, T], L^2(\mathbb{R})^m)$ that solves $(CNLS)_p$ on $[-T, T]$,
- u is the unique function in the space $\mathbf{S}([-T, T] \times \mathbb{R})^m$ that solves $(CNLS)_p$,
- the solution depends continuously on the initial data.

In this Theorem, we only state uniqueness of the solution in the Strichartz space $\mathbf{S}([-T, T] \times \mathbb{R})^m$. It is not known whether the uniqueness result can be extended to $C([-T, T], L^2(\mathbb{R})^m)$. Once local existence is established, the uniqueness for small time allows us to define the maximal solution u on the interval $[-\underline{T}, \bar{T}]$ where

$$\bar{T} = \sup\{T > 0, \text{ such that } (CNLS)_p \text{ has a solution on } [0, T]\},$$

$$\underline{T} = \sup\{T > 0, \text{ such that } (CNLS)_p \text{ has a solution on } [-T, 0]\}.$$

Our aim now is to establish that maximal solutions for the problem $(CNLS)_p$ with initial data in $L^2(\mathbb{R})^m$ are global solutions. We need the following results, the first one

is a classical result concerning conservation of the mass, the second one is a blow up alternative.

Lemma 4. *Let $u \in C([-T, T], L^2(\mathbb{R}))^m$ be a local solution of the Cauchy problem $(CNLS)_p$ with initial data $\psi \in L^2(\mathbb{R})^m$, then*

$$\|u_j(t, \cdot)\|_2 = \|\psi_j(t, \cdot)\|_2, \quad \forall t \in [-T, T]. \quad (3)$$

Lemma 5. *Assume hypothesis of Theorem 2. If $\bar{T} < \infty$, so there is j such that*

$$\lim_{t \rightarrow \bar{T}} \|u_j(t, \cdot)\|_{L^2(\mathbb{R})} = +\infty.$$

Analogous result holds if $\underline{T} < \infty$.

Using these results, the maximal solution is actually global.

Corollary 1. *Assume hypothesis of Theorem 2. The Cauchy problem has a unique global solution $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R})^m$.*

3. Tools

Let recall that using some properties of the Fourier transform, we can rewrite the free Shrödinger operator $S(t)f = e^{it\Delta}f$ as following:

$$S(t)f = \mathfrak{F}^{-1}(\exp(-it|\cdot|^2)) * f.$$

Then, we summarize some of its properties in the following Lemma:

Lemma 6. *We have:*

- $S(t)$ is an isometry of $L^2(\mathbb{R})$,
- the adjoint of the operator $S(t)$ is $S(t)^* = S(-t)$,
- Duhamel formula: $S(t)\psi - i\gamma \int_0^t S(t-s)|u|^{p-1}uds$ is the solution to the problem $(NLS)_p$.

In order to prove local existence for the nonlinear coupled Shrödinger equations, we need some space-time estimates.

Theorem 3. *(Strichartz estimates, see [2, 8])*

Let (q, r) and (q_1, r_1) be two admissible pairs. Denoting by q'_1 and r'_1 the dual exponents for q_1 and r_1 . Then, the following estimates hold:

$$\|S(t)f\|_{L^q L^r(\mathbb{R} \times \mathbb{R})} \leq C(q)\|f\|_2, \quad (4)$$

$$\left\| \int_0^t S(t-s)F(s)ds \right\|_{L^q L^r(\mathbb{R} \times \mathbb{R})} \leq C(q, q_1)\|F\|_{L^{q'_1} L^{r'_1}(\mathbb{R} \times \mathbb{R})}, \quad (5)$$

for some different positive constants $C(q)$ and $C(q, q_1)$.

In order to prove Lemma 1, we make use of the following Lemma:

Lemma 7. (See [9])

For any $a, b \in \mathbb{C}$ and $p \geq 1$, we have

$$\left| |a|^{p-1}a - |b|^{p-1}b \right| \leq 2p\{|a|^{p-1} + |b|^{p-1}\}|a - b|.$$

Results in Lemma 2 and Lemma 3 need some Banach algebra properties.

Lemma 8. (See [15])

(1) If $s > 0$, then for all $f, g \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ the following Leibniz rule holds

$$\|fg\|_{H^s} \leq C_1(s)\{\|f\|_{H^s}\|g\|_\infty + \|f\|_\infty\|g\|_{H^s}\}.$$

(2) If $s > \frac{1}{2}$, then we have

$$\|fg\|_{H^s} \leq C_2(s)\|f\|_{H^s}\|g\|_{H^s},$$

where $C_1(s)$ and $C_2(s)$ are two convenient positive constants.

Lemma 9. (See [9])

Let $s > \frac{1}{2}$ be a real number and $p \geq 1$ be an odd integer. Then for all $u, v \in H^s(\mathbb{R})$ we have

$$\left\| |u|^{p-1}u - |v|^{p-1}v \right\|_{H^s} \leq pC_2(s)^{p-1}\{\|u\|_{H^s}^{p-1} + \|v\|_{H^s}^{p-1}\}\|u - v\|_{H^s}.$$

Especially here, in order to determine correctly the constant which is not claimed in [9], we repeat its proof. When $p = 1$, it is obvious that inequality holds. Assume $p > 1$, we have

$$\begin{aligned} \left\| |u|^{p-1}u - |v|^{p-1}v \right\|_{H^s} &= \left\| u^{\frac{p+1}{2}} \bar{u}^{\frac{p-1}{2}} - v^{\frac{p+1}{2}} \bar{v}^{\frac{p-1}{2}} \right\|_{H^s} \\ &\leq \left\| u^{\frac{p+1}{2}} \bar{u}^{\frac{p-1}{2}} - u^{\frac{p+1}{2}} \bar{u}^{\frac{p-3}{2}} \bar{v} \right\|_{H^s} + \left\| u^{\frac{p+1}{2}} \bar{u}^{\frac{p-3}{2}} \bar{v} - u^{\frac{p+1}{2}} \bar{u}^{\frac{p-5}{2}} \bar{v}^2 \right\|_{H^s} \\ &+ \dots + \left\| u^{\frac{p+1}{2}} \bar{u} \bar{v}^{\frac{p-3}{2}} - u^{\frac{p+1}{2}} \bar{v}^{\frac{p-1}{2}} \right\|_{H^s} + \left\| u^{\frac{p+1}{2}} \bar{v}^{\frac{p-1}{2}} - v^{\frac{p+1}{2}} \bar{v}^{\frac{p-1}{2}} \right\|_{H^s} \\ &\leq C_2(s)^{p-1} \|u - v\|_{H^s} \left\{ \sum_{k=0}^{p-1} \|u\|_{H^s}^{p-1-k} \|v\|_{H^s}^k \right\}. \end{aligned}$$

Thanks to the Young's inequality, we get

$$\|u\|_{H^s}^{p-1-k} \|v\|_{H^s}^k \leq \frac{p-1-k}{p-1} \|u\|_{H^s}^{p-1} + \frac{k}{p-1} \|v\|_{H^s}^{p-1}.$$

Then

$$\left\| |u|^{p-1}u - |v|^{p-1}v \right\|_{H^s} \leq pC_2(s)^{p-1}\{\|u\|_{H^s}^{p-1} + \|v\|_{H^s}^{p-1}\}\|u - v\|_{H^s}.$$

Now we state an other important result which concerns the fixed point theorem used to prove local well-posedness for our nonlinear problem.

Theorem 4. (Iteration Principle, See [9])

Let \mathbf{E} and \mathbf{F} be Banach spaces. For any operator $\mu : \mathbf{F} \rightarrow \mathbf{E}$ and $\nu : \mathbf{E} \rightarrow \mathbf{F}$ which satisfies:

- the operator μ is linear and bounded,

$$\forall f \in \mathbf{F}, \|\mu f\|_{\mathbf{E}} \leq C\|f\|_{\mathbf{F}}, \quad (6)$$

- the operator ν needs not to be linear, $\nu(0) = 0$ and for some $r > 0$ we have

$$\forall f, g \in B_{2r} = \{h \in \mathbf{E}, \|h\|_{\mathbf{E}} \leq 2r\}, \|\nu(f) - \nu(g)\|_{\mathbf{F}} \leq \frac{1}{2C}\|f - g\|_{\mathbf{E}}. \quad (7)$$

Then

- for all $\phi \in B_r$, there is a unique solution $u \in B_{2r}$ to the equation

$$u = \phi + \mu\nu(u), \quad (8)$$

- if $v \in B_{2r}$ solves the equation $v = \phi + \mu\nu(v)$ where $\phi \in B_r$, then we have

$$\|u - v\|_{\mathbf{E}} \leq 2\|\phi - \varphi\|_{\mathbf{E}}. \quad (9)$$

4. Local well-posedness for regular data

At first, we are going to prove two lemma mentioned above.

4.1. Proof of Lemma 1

Note that if we are in one of these cases $a = 0, b = 0, \alpha = 0$ or $\beta = 0$, then the desired result is obvious. For example and without losing generality, when $a = 0$, we have

$$\||\alpha|^p|\beta|^{p-2}\beta| = |\alpha||\alpha|^{p-1}|\beta|^{p-1} \leq \frac{1}{2}\{|\alpha|^{2(p-1)} + |\beta|^{2(p-1)}\}(|\alpha - 0| + |\beta - b|).$$

In the case $|a| = |b|$ and $|\alpha| = |\beta|$, from Lemma 7 we obtain

$$\||b|^{2(p-1)}b - |\beta|^{2(p-1)}\beta| \leq 2(2p - 1)\{|b|^{2(p-1)} + |\beta|^{2(p-1)}\}|b - \beta|.$$

This implies the desired inequality. Now and without losing generality, we treat just the following cases.

First case: Assuming that $|a| \geq |\alpha|$ and $|b| \geq |\beta|$, we have

$$\||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| = |a|^p|b|^{p-1}|1 - |x|^p|y|^{p-2}y|, \quad (10)$$

where $x = \frac{\alpha}{a}$ and $y = \frac{\beta}{b}$. Using the triangle inequality, we get

$$|1 - |x|^p|y|^{p-2}y| \leq |1 - |x|^p| + |x|^p|1 - |y|^{p-2}y|. \quad (11)$$

Using Lemma 7, one can write

$$|1 - |y|^{p-2}y| \leq 2(p-1)|1 - y|(1 + |y|^{p-2}). \tag{12}$$

At first, we assume that p is an integer. Since $|x| \leq 1$, we have

$$|1 - |x|^p| \leq |1 - x| \sum_{k=0}^{p-1} |x|^k \leq p|1 - x|. \tag{13}$$

Then, equation (11) becomes

$$|1 - |x|^p|y|^{p-2}y| \leq p|1 - x| + 2(p-1)|x|^p|1 - y|(1 + |y|^{p-2}),$$

Combining with (10), we get

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq |a|^p|b|^{p-1}\{p|1 - x| + 2(p-1)|x|^p|1 - y|(1 + |y|^{p-2})\}.$$

Replacing x and y , it derives that

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq p|a|^{p-1}|b|^{p-1}|a - \alpha| + 2(p-1)\{|\alpha|^p|b|^{p-2} + |\alpha|^p|\beta|^{p-2}\}|b - \beta|.$$

Then, we obtain

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq 2p\{|a|^{p-1}|b|^{p-1} + |\alpha|^p|b|^{p-2} + |\alpha|^p|\beta|^{p-2}\}(|a - \alpha| + |b - \beta|). \tag{14}$$

By using Young's inequality, this yields to

$$|a|^{p-1}|b|^{p-1} \leq \frac{1}{2}\{|a|^{2(p-1)} + |b|^{2(p-1)}\},$$

$$|\alpha|^p|b|^{p-2} \leq \frac{p}{2(p-1)}|\alpha|^{2(p-1)} + \frac{p-2}{2(p-1)}|b|^{2(p-1)} \leq \frac{1}{2}|\alpha|^{2(p-1)} + \frac{1}{2}|b|^{2(p-1)},$$

$$|\alpha|^p|\beta|^{p-2} \leq \frac{p}{2(p-1)}|\alpha|^{2(p-1)} + \frac{p-2}{2(p-1)}|\beta|^{2(p-1)} \leq \frac{1}{2}|\alpha|^{2(p-1)} + \frac{1}{2}|\beta|^{2(p-1)}.$$

Then (14) gives the desired result but only for integer exponent

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq 2p\{|a|^{2(p-1)} + |b|^{2(p-1)} + |\alpha|^{2(p-1)} + |\beta|^{2(p-1)}\}(|a - \alpha| + |b - \beta|).$$

Else if p is not an integer, then we introduce $[p]$ the whole part of p . Since $|x| \leq 1$, this yields to $|x|^{[p]+1} \leq |x|^p \leq 1$ and then comparing distance between the point 1 and the other points $|x|^p$ and $|x|^{[p]+1}$, we get

$$|1 - |x|^p| \leq |1 - |x|^{[p]+1}|.$$

Using inequality (13), we obtain

$$|1 - |x|^p| \leq ([p] + 1)|1 - x|. \quad (15)$$

Now, by combining inequalities (12) and (15) with (11) we obtain

$$\begin{aligned} |1 - |x|^p|y|^{p-2}y| &\leq ([p] + 1)|1 - x| + 2(p-1)|x|^p|1 - y|(1 + |y|^{p-2}) \\ &\leq 2p\{|1 - x| + |x|^p|1 - y|(1 + |y|^{p-2})\}. \end{aligned}$$

Replacing x by $\frac{\alpha}{a}$ and y by $\frac{\beta}{b}$, it follows that

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq 2p\{|a|^{p-1}|b|^{p-1} + |\alpha|^p|b|^{p-2} + |\alpha|^p|\beta|^{p-2}\}(|a - \alpha| + |b - \beta|).$$

Thanks to the Young's inequality, we deduce that

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq 2p\{|a|^{2(p-1)} + |b|^{2(p-1)} + |\alpha|^{2(p-1)} + |\beta|^{2(p-1)}\}(|a - \alpha| + |b - \beta|).$$

Second case: Here, we suppose that $|a| \geq |\alpha|$ and $|b| \leq |\beta|$, let $x = \frac{\alpha}{a}$ and $y = \frac{b}{\beta}$, we have

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| = |a|^p|\beta|^{p-1}||y|^{p-2}y - |x|^p|.$$

So, by using the triangle inequality, we get

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq |a|^p|\beta|^{p-1}\{||y|^{p-2}y - 1| + |1 - |x|^p|\}. \quad (16)$$

As above we distinguish two cases for the number p . In the case where p is an integer, by combining (12) and (13) with (16), it follows that

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq 2p|a|^p|\beta|^{p-1}\{1 - y(1 + |y|^{p-2}) + |1 - x|\}.$$

Replacing x by $\frac{\alpha}{a}$ and y by $\frac{b}{\beta}$, it derives that

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq 2p\{|a|^{p-1}|\beta|^{p-1} + |a|^p|\beta|^{p-2} + |a|^p|b|^{p-2}\}(|a - \alpha| + |b - \beta|).$$

By using Young's inequality as above, we obtain

$$||a|^p|b|^{p-2}b - |\alpha|^p|\beta|^{p-2}\beta| \leq 2p\{|a|^{2(p-1)} + |b|^{2(p-1)} + |\alpha|^{2(p-1)} + |\beta|^{2(p-1)}\}(|a - \alpha| + |b - \beta|).$$

Finally, in the case where p is not an integer, we introduce the whole part $[p]$ and then by interpolating we obtain our desired inequality.

4.2. Proof of Lemma 2

Let $s > \frac{1}{2}$ and $u, v \in H^s(\mathbb{R})$. Taking $q \in \mathbb{N}^*$, we introduce a suitable telescope sum and then by means of the Minkowski inequality, we obtain

$$\|u^q - v^q\|_{H^s} \leq \|u^q - u^{q-1}v\|_{H^s} + \|u^{q-1}v - u^{q-2}v^2\|_{H^s} + \dots + \|uv^{q-1} - v^q\|_{H^s}.$$

From Lemma 8, we have for all integer k such that $0 \leq k \leq q - 1$

$$\|u^{q-k}v^k - u^{q-k-1}v^{k+1}\|_{H^s} = \|u^{q-k-1}v^k(u-v)\|_{H^s} \leq C_2(s)^{q-1} \|u\|_{H^s}^{q-k-1} \|v\|_{H^s}^k \|u-v\|_{H^s}.$$

Immediately, this yields to

$$\|u^q - v^q\|_{H^s} \leq C_2(s)^{q-1} \|u - v\|_{H^s} \sum_{k=0}^{q-1} \|u\|_{H^s}^{q-k-1} \|v\|_{H^s}^k.$$

This implies the desired result by taking $C(s, q) = C_2(s)^{q-1}$.

4.3. Proof of Lemma 3

We first remind that $s > \frac{1}{2}$ and $p \geq 2$ is an even integer. Let $u, v, f, g \in H^s(\mathbb{R})$, we have

$$\| |u|^p |v|^{p-2} v - |f|^p |g|^{p-2} g \|_{H^s} = \| u^{\frac{p}{2}} v^{\frac{p}{2}} \bar{u}^{\frac{p}{2}} \bar{v}^{\frac{p-2}{2}} - f^{\frac{p}{2}} g^{\frac{p}{2}} \bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}} \|_{H^s}.$$

When $p = 2$, we have

$$\| |u|^2 v - |f|^2 g \|_{H^s} = \| u\bar{u}v - f\bar{f}g \|_{H^s} \leq \| u\bar{u}v - u\bar{u}g \|_{H^s} + \| u\bar{u}g - u\bar{f}g \|_{H^s} + \| u\bar{f}g - f\bar{f}g \|_{H^s}.$$

Applying Lemma 8, we obtain

$$\| |u|^2 v - |f|^2 g \|_{H^s} \leq C_2(s)^2 \{ \|u\|_{H^s}^2 \|v-g\|_{H^s} + \|u\|_{H^s} \|g\|_{H^s} \|u-f\|_{H^s} + \|f\|_{H^s} \|g\|_{H^s} \|u-f\|_{H^s} \}.$$

So

$$\| |u|^2 v - |f|^2 g \|_{H^s} \leq \frac{3}{2} C_2(s)^2 \{ \|u\|_{H^s}^2 + \|v\|_{H^s}^2 + \|f\|_{H^s}^2 + \|g\|_{H^s}^2 \} (\|u-f\|_{H^s} + \|v-g\|_{H^s}). \tag{17}$$

Else if $p \geq 4$, we introduce four suitable terms, then by applying the Minkowski inequality, it yields to

$$\| u^{\frac{p}{2}} v^{\frac{p}{2}} \bar{u}^{\frac{p}{2}} \bar{v}^{\frac{p-2}{2}} - f^{\frac{p}{2}} g^{\frac{p}{2}} \bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}} \|_{H^s} \leq \| (u^{\frac{p}{2}} v^{\frac{p}{2}} \bar{u}^{\frac{p}{2}}) \bar{v}^{\frac{p-2}{2}} - (u^{\frac{p}{2}} v^{\frac{p}{2}} \bar{u}^{\frac{p}{2}}) \bar{g}^{\frac{p-2}{2}} \|_{H^s} + \tag{18}$$

$$\| (\bar{g}^{\frac{p-2}{2}} u^{\frac{p}{2}} v^{\frac{p}{2}}) \bar{u}^{\frac{p}{2}} - (\bar{g}^{\frac{p-2}{2}} u^{\frac{p}{2}} v^{\frac{p}{2}}) \bar{f}^{\frac{p}{2}} \|_{H^s} + \| (\bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}} u^{\frac{p}{2}}) v^{\frac{p}{2}} - (\bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}} u^{\frac{p}{2}}) g^{\frac{p}{2}} \|_{H^s} +$$

$$\| (g^{\frac{p}{2}} \bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}}) u^{\frac{p}{2}} - (g^{\frac{p}{2}} \bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}}) f^{\frac{p}{2}} \|_{H^s}.$$

Now, we treat separately each term of the second member of the previous inequality. Thus, using Lemma 8 and Lemma 2, this yields to

$$\begin{aligned} \|(u^{\frac{p}{2}}v^{\frac{p}{2}}\bar{u}^{\frac{p}{2}})\bar{v}^{\frac{p-2}{2}} - (u^{\frac{p}{2}}v^{\frac{p}{2}}\bar{u}^{\frac{p}{2}})\bar{g}^{\frac{p-2}{2}}\|_{H^s} &\leq C_2(s)\|u^{\frac{p}{2}}v^{\frac{p}{2}}\bar{u}^{\frac{p}{2}}\|_{H^s}\|\bar{v}^{\frac{p-2}{2}} - \bar{g}^{\frac{p-2}{2}}\|_{H^s} \\ &\leq C_2(s)^{2(p-1)}\|u\|_{H^s}^p\|v - g\|_{H^s}\sum_{k=0}^{\frac{p-4}{2}}\|v\|_{H^s}^{(p-2)-k}\|g\|_{H^s}^k. \end{aligned}$$

By Young’s inequality, we have

$$\|v\|_{H^s}^{(p-2)-k}\|g\|_{H^s}^k \leq \frac{(p-2) - k}{p-2}\|v\|_{H^s}^{p-2} + \frac{k}{p-2}\|g\|_{H^s}^{p-2} \leq \|v\|_{H^s}^{p-2} + \|g\|_{H^s}^{p-2}.$$

So

$$\begin{aligned} \|(u^{\frac{p}{2}}v^{\frac{p}{2}}\bar{u}^{\frac{p}{2}})\bar{v}^{\frac{p-2}{2}} - (u^{\frac{p}{2}}v^{\frac{p}{2}}\bar{u}^{\frac{p}{2}})\bar{g}^{\frac{p-2}{2}}\|_{H^s} &\leq \frac{p-2}{2}C_2(s)^{2(p-1)}\{\|u\|_{H^s}^p\|v\|_{H^s}^{p-2} + \|u\|_{H^s}^p\|g\|_{H^s}^{p-2}\} \\ &\quad \|v - g\|_{H^s}. \end{aligned}$$

Then, using Young’s inequality against, we obtain

$$\begin{aligned} \|(u^{\frac{p}{2}}v^{\frac{p}{2}}\bar{u}^{\frac{p}{2}})\bar{v}^{\frac{p-2}{2}} - (u^{\frac{p}{2}}v^{\frac{p}{2}}\bar{u}^{\frac{p}{2}})\bar{g}^{\frac{p-2}{2}}\|_{H^s} &\leq \frac{p-2}{2}C_2(s)^{2(p-1)}\{\|u\|_{H^s}^{2(p-1)} + \|v\|_{H^s}^{2(p-1)} \\ &\quad + \|g\|_{H^s}^{2(p-1)}\}\|v - g\|_{H^s}. \end{aligned} \tag{19}$$

With any loss of generality, we expose the method just for the second term from the right member of the inequality (18) and treat others in the same way. Using Lemma 8, we have

$$\|(\bar{g}^{\frac{p-2}{2}}u^{\frac{p}{2}}v^{\frac{p}{2}})\bar{u}^{\frac{p}{2}} - (\bar{g}^{\frac{p-2}{2}}u^{\frac{p}{2}}v^{\frac{p}{2}})\bar{f}^{\frac{p}{2}}\|_{H^s} \leq C_2(s)^{\frac{3p-2}{2}}\|g\|_{H^s}^{\frac{p-2}{2}}\|u\|_{H^s}^{\frac{p}{2}}\|v\|_{H^s}^{\frac{p}{2}}\|\bar{u}^{\frac{p}{2}} - \bar{f}^{\frac{p}{2}}\|_{H^s}.$$

Together Lemma 8, Lemma 2 and Young’s inequality, give

$$\begin{aligned} \|(\bar{g}^{\frac{p-2}{2}}u^{\frac{p}{2}}v^{\frac{p}{2}})\bar{u}^{\frac{p}{2}} - (\bar{g}^{\frac{p-2}{2}}u^{\frac{p}{2}}v^{\frac{p}{2}})\bar{f}^{\frac{p}{2}}\|_{H^s} &\leq C_2(s)^{2(p-1)}\|g\|_{H^s}^{\frac{p-2}{2}}\|v\|_{H^s}^{\frac{p}{2}}\|u - f\|_{H^s} \\ &\quad \sum_{k=0}^{\frac{p-2}{2}}\|u\|_{H^s}^{(p-1)-k}\|f\|_{H^s}^k \\ &\leq \frac{p}{2}C_2(s)^{2(p-1)}\|g\|_{H^s}^{\frac{p-2}{2}}\|v\|_{H^s}^{\frac{p}{2}}\|u - f\|_{H^s} \\ &\quad \{\|u\|_{H^s}^{p-1} + \|f\|_{H^s}^{p-1}\}. \end{aligned}$$

Remarking that $\frac{p-2}{2} + \frac{p}{2} + (p-1) = 2(p-1)$, then it is possible to write Young’s inequality with three terms

$$\|g\|_{H^s}^{\frac{p-2}{2}}\|v\|_{H^s}^{\frac{p}{2}}\|u\|_{H^s}^{p-1} \leq \frac{p-2}{4(p-1)}\|g\|_{H^s}^{2(p-1)} + \frac{p}{4(p-1)}\|v\|_{H^s}^{2(p-1)} + \frac{1}{2}\|u\|_{H^s}^{2(p-1)},$$

and

$$\|g\|_{H^s}^{\frac{p-2}{2}} \|v\|_{H^s}^{\frac{p}{2}} \|f\|_{H^s}^{p-1} \leq \frac{p-2}{4(p-1)} \|g\|_{H^s}^{2(p-1)} + \frac{p}{4(p-1)} \|v\|_{H^s}^{2(p-1)} + \frac{1}{2} \|f\|_{H^s}^{2(p-1)}.$$

So

$$\begin{aligned} \|g\|_{H^s}^{\frac{p-2}{2}} \|v\|_{H^s}^{\frac{p}{2}} \|u\|_{H^s}^{p-1} + \|g\|_{H^s}^{\frac{p-2}{2}} \|v\|_{H^s}^{\frac{p}{2}} \|f\|_{H^s}^{p-1} &\leq \frac{1}{2} \{ \|u\|_{H^s}^{2(p-1)} + \|v\|_{H^s}^{2(p-1)} + \|f\|_{H^s}^{2(p-1)} \\ &\quad + \|g\|_{H^s}^{2(p-1)} \}. \end{aligned}$$

Then

$$\|(\bar{g}^{\frac{p-2}{2}} u^{\frac{p}{2}} v^{\frac{p}{2}}) \bar{u}^{\frac{p}{2}} - (\bar{g}^{\frac{p-2}{2}} u^{\frac{p}{2}} v^{\frac{p}{2}}) \bar{f}^{\frac{p}{2}}\|_{H^s} \leq \frac{p}{4} C_2(s)^{2(p-1)} \left\{ \sum_{h \in \{u,v,f,g\}} \|h\|_{H^s}^{2(p-1)} \right\} \|u - f\|_{H^s}.$$

Now, from (19) and analogous inequalities, when combined with (18), this yields to

$$\|u^{\frac{p}{2}} v^{\frac{p}{2}} \bar{u}^{\frac{p}{2}} \bar{v}^{\frac{p-2}{2}} - f^{\frac{p}{2}} g^{\frac{p}{2}} \bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}}\|_{H^s} \leq \frac{p}{2} C_2(s)^{2(p-1)} \left\{ \sum_{h \in \{u,v,f,g\}} \|h\|_{H^s}^{2(p-1)} \right\} (\|u - f\|_{H^s} + \|v - g\|_{H^s}).$$

Since we have

$$\| |u|^p |v|^{p-2} v - |f|^p |g|^{p-2} g \|_{H^s} = \| u^{\frac{p}{2}} v^{\frac{p}{2}} \bar{u}^{\frac{p}{2}} \bar{v}^{\frac{p-2}{2}} - f^{\frac{p}{2}} g^{\frac{p}{2}} \bar{f}^{\frac{p}{2}} \bar{g}^{\frac{p-2}{2}} \|_{H^s},$$

taking into account the inequality (17), then we obtain our desired inequality by taking

$$C(s, p) = \frac{3p}{2} C_2(s)^{2(p-1)}, \text{ for all even integer } p \geq 2.$$

4.4. Proof of Theorem 1

We use standard fixed point arguments cited in Theorem 4. For $T > 0$, let $\mathbf{E} = \mathbf{F} = C([-T, T], H^s(\mathbb{R}))^m$. Fix $R > 0$, we prove the existence of some small $T > 0$ which allows us to apply the iteration principle with the suitable space \mathbf{E} . We consider the operator $\mu = (\mu_j) : \mathbf{F} \rightarrow \mathbf{E}$, each of its components is defined for all $f = (f_j) \in \mathbf{F}$ as follows

$$\mu_j f_j(t, x) = -i \int_0^t S(t - \tau) f_j(\tau, x) d\tau, \quad (t, x) \in [-T, T] \times \mathbb{R}.$$

Being an integral, the operator μ is linear and continuous in time. Taking the H^s -norm, we obtain

$$\begin{aligned} \|\mu_j f_j(t, \cdot)\|_{H^s}^2 &= \|(1 + |\xi|^2)^{\frac{s}{2}} \mathfrak{F}[\mu_j f_j](\xi)\|_2^2 \\ &= \int_{\mathbb{R}} \left| \int_0^t (1 + |\xi|^2)^{\frac{s}{2}} \exp(-i|\xi|^2(t - \tau)) \mathfrak{F}[f_j(\tau, \cdot)](\xi) d\tau \right|^2 d\xi. \end{aligned}$$

By using the Jensen inequality we obtain

$$\|\mu_j f_j(t, \cdot)\|_{H^s}^2 \leq |t| \int_{\mathbb{R}} \left| \int_0^t (1 + |\xi|^2)^s |\mathfrak{F}[f_j(\tau, \cdot)](\xi)|^2 d\tau \right| d\xi.$$

Thanks to the Fubini theorem, we get

$$\|\mu_j f_j(t, \cdot)\|_{H^s}^2 \leq |t| \int_0^t \|f_j(\tau, \cdot)\|_{H^s}^2 d\tau.$$

To check that μf satisfies the bound inequality (6), we use the standard norm $\sup_j (\sup_{t \in [-T, T]} \|\cdot\|_{H^s})$ which equipped the space $C([-T, T], H^s(\mathbb{R}))^m$. Then, we get

$$\|\mu f\|_{\mathbf{E}} \leq T \|f\|_{\mathbf{F}}. \quad (20)$$

Let $\psi = (\psi_j)_j \in H^s(\mathbb{R})^m$ and $\phi = (\phi_j)_j$ such that $\phi_j(t, \cdot) = S(t)\psi_j(\cdot)$ be given by the first term of the Duhamel's formula. Assume that $\|\psi_j\|_{H^s} \leq R$, by using the inversion Fourier formula, we have

$$\mathfrak{F}[\phi_j](t, x) = \exp(-i|x|^2 t) \mathfrak{F}[\psi_j](t, x).$$

It follows that for all $t \in [-T, T]$

$$\|\phi_j(t, \cdot)\|_{H^s} = \|\psi_j(t, \cdot)\|_{H^s} \leq R. \quad (21)$$

Now, let $\nu = (\nu_j)_j : \mathbf{E} \rightarrow \mathbf{F}$ be defined by

$$\nu_j(f) = \gamma \sum_{k=1}^m |f_k|^p |f_j|^{p-2} f_j, \forall f = (f_j) \in \mathbf{E}.$$

It is obvious that $\nu(0) = 0$. Let $u = (u_j)_j$ and $v = (v_j)_j$ be in the ball of \mathbf{E} centered on 0 with radius $2R$. We have

$$\|\nu(u) - \nu(v)\|_{\mathbf{E}} = \sup_j (\sup_{t \in [-T, T]} \|\nu_j(u) - \nu_j(v)\|_{H^s}).$$

Then, by using the triangle inequality with the H^s -norm, Lemma 9 and Lemma 3, we get

$$\begin{aligned} \|\nu_j(u) - \nu_j(v)\|_{H^s} &\leq \gamma \{ \| |u_j|^{2(p-1)} u_j - |v_j|^{2(p-1)} v_j \|_{H^s} + \sum_{k=1, k \neq j}^m \| |u_k|^p |u_j|^{p-2} u_j \\ &\quad - |v_k|^p |v_j|^{p-2} v_j \|_{H^s} \}. \end{aligned}$$

Since we have $(2p-1) - \frac{3}{2}p = \frac{p-2}{2} \geq 0$, then

$$\begin{aligned} \|\nu_j(u) - \nu_j(v)\|_{H^s} &\leq \gamma (2p-1) C_2(s)^{2(p-1)} \sum_{k=1}^m \left\{ \sum_{h \in \{u_k, u_j, v_k, v_j\}} \|h\|_{H^s}^{2(p-1)} \right\} (\|u_k - v_k\|_{H^s} \\ &\quad + \|u_j - v_j\|_{H^s}). \end{aligned}$$

Then

$$\|\nu(u) - \nu(v)\|_{\mathbf{F}} \leq 4m\gamma(2p-1)C_2(s)^{2(p-1)} \{ \|u\|_{\mathbf{E}}^{2(p-1)} + \|v\|_{\mathbf{E}}^{2(p-1)} \} \|u - v\|_{\mathbf{E}}.$$

This yields to

$$\|\nu(u) - \nu(v)\|_{\mathbf{F}} \leq 8m\gamma(2p - 1)C_2(s)^{2(p-1)}(2R)^{2(p-1)}\|u - v\|_{\mathbf{E}}. \tag{22}$$

Consequently, let $T > 0$ such that $\frac{1}{2T} = 8m\gamma(2p - 1)C_2(s)^{2(p-1)}(2R)^{2(p-1)}$. Applying the iteration principle given by Theorem 4 with constant $C := T$, this allows us to construct a solution $u \in B_{2R}$ for the Cauchy problem $(CNLS)_p$.

Now, we have to prove the uniqueness of a solution to our problem. Let $\tau > 0$ and $u, v \in \mathbf{E} = C([- \tau, \tau], H^s(\mathbb{R}))^m$ be two solutions of our system, then we define $R = \max(\|u\|_{\mathbf{E}}, \|v\|_{\mathbf{E}})$ and $w := u - v$, we have

$$\begin{cases} i\partial_t w_j + \Delta w_j = \gamma \sum_{k=1}^m (|u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j), t \in [-\tau, \tau], \\ w_j(0, x) = 0, x \in \mathbb{R}. \end{cases}$$

Thanks to the Duhamel form, one have for all $t \in [-T, T]$

$$w_j(t, \cdot) = \mu_j \sum_{k=1}^m (|u_k|^p |u_j|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j) = \mu_j (\nu_j(u) - \nu_j(v)).$$

Using previous computations which allows us to (20), we get

$$\|w\|_{\mathbf{E}} \leq \tau \|\nu(u) - \nu(v)\|_{\mathbf{E}}.$$

Thanks to inequality (22), we obtain

$$\|\nu(u) - \nu(v)\|_{\mathbf{F}} \leq 8m\gamma(2p - 1)C_2(s)^{2(p-1)}(2R)^{2(p-1)}\|w\|_{\mathbf{E}}.$$

Fix $T > 0$ such that $\frac{1}{2T} = 8m\gamma(2p - 1)C_2(s)^{2(p-1)}(2R)^{2(p-1)}$, then two cases occur.

If $\tau \leq T$, then for all $t \in [-\tau, \tau]$, we have $\|w\|_{\mathbf{E}} \leq \frac{1}{2}\|w\|_{\mathbf{E}}$. This implies that $u = v$ on $[-\tau, \tau]$. Else, if $T < \tau$, then and in a same way as previous we prove that $\|w\|_{\mathbf{E}} \leq \frac{1}{2}\|w\|_{\mathbf{E}}$ but on $[-T, T] \subset [-\tau, \tau]$. This implies that $u = v$ on $[-T, T]$ and then uniqueness for all time $[-\tau, \tau]$ will be produced by a translation argument. Here it is important to remark that R , consequently T , do not change during all necessary translation.

To finish the proof of our Theorem 1, we prove the continuous dependence from initial data. Taking $R > 0$, let $\psi, \varphi \in B_R \subset \mathbf{E}$ and $u, v \in B_{2R} \subset \mathbf{E}$ be the solutions of our problem produced by the iteration method with initial data ψ, φ respectively. By using result (9) we obtain

$$\|u - v\|_{\mathbf{E}} \leq 2\|S(\cdot)\psi - S(\cdot)\varphi\|_{\mathbf{E}}.$$

This, together with (21), give us

$$\|u_1 - u_2\|_{\mathbf{E}} \leq 2\|\psi - \varphi\|_{\mathbf{E}},$$

which produce obviously the continuity with respect to initial data. Thus, our Cauchy problem is actually local well-posed.

4.5. Proof of Theorem 2

Let (q, r) admissible, using Hölder inequality we have

$$\|g\|_{\mathbf{N}(I \times \mathbb{R})} = \sup_{f \in \mathbf{S}(I \times \mathbb{R}), \|f\|_{\mathbf{S}(I \times \mathbb{R})} \leq 1} \int_{I \times \mathbb{R}} g(t, x) \cdot f(t, x) dt dx \leq \|g\|_{L^{q'} L^{r'}(I \times \mathbb{R})},$$

where (q', r') is the dual exponent of (q, r) . Let $F_j(u) = \sum_{k=1}^m |u_k|^p |u_j|^{p-2} u_j$, writing the usual Duhamel form

$$u_j(t, \cdot) = S(t)\psi_j(\cdot) - i \int_0^t S(t-s)F_j(u)(s, \cdot) ds.$$

At the sequel, we are going to prove our Theorem by means of the iteration principle summarized in Theorem 4. Hence, we define spaces $\mathbf{E} := \mathbf{S}(I \times \mathbb{R})^m$ and $\mathbf{F} := \mathbf{N}(I \times \mathbb{R})^m$, we define also operator $\mu : \mathbf{F} \rightarrow \mathbf{E}$ and $\nu : \mathbf{E} \rightarrow \mathbf{F}$ as in the proof of Theorem 1. Taking $R > 0$ and $\psi \in (L^2(\mathbb{R}))^m$ such that $\|\psi_j\|_2 \leq R$, using the Strichartz inequality (4) with the unified constant $C_1 = \sup_{q \in [2, +\infty]} C(q)$ we get

$$\|S(t)\psi_j\|_{\mathbf{S}(I \times \mathbb{R})} \leq C_1 \|\psi_j\|_2 \leq C_1 R. \quad (23)$$

Let us remark that C_1 does not depend on the interval $I = [-T, T]$ which we will choose in the next part of this proof. By using definition of the \mathbf{N} -norm and Strichartz inequality (5) we have

$$\|\mu_j F_j\|_{\mathbf{S}(I \times \mathbb{R})} \leq C_1 \|F_j\|_{\mathbf{N}(I \times \mathbb{R})}.$$

We take $u, v \in B_{2C_1 R} \subset \mathbf{E}$. We make use of $(q, r) = (\frac{4p}{p-1}, 2p)$, since we have $p \in [2, 3[$ then $2 < r = 2p < q < +\infty$. Also, we have $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$, meaning that (q, r) is admissible. Let us remark that

$$r' = \frac{r}{r-1} = \frac{2p}{2p-1}.$$

Now, we are going to estimate the $L^{r'}(\mathbb{R})$ norm of $\nu_j(u) - \nu_j(v)$. By using the triangle inequality we have:

$$\begin{aligned} \|\nu_j(u) - \nu_j(v)\|_{r'} &\leq \gamma \{ \| |u_j|^{2(p-1)} u_j - |v_j|^{2(p-1)} v_j \|_{r'} + \sum_{k=1, k \neq j}^m \| |u_k|^p |u_j|^{p-2} u_j \\ &\quad - |v_k|^p |v_j|^{p-2} v_j \|_{r'} \}. \end{aligned}$$

From Lemma 1 and Lemma 7, this yields to

$$\|\nu_j(u) - \nu_j(v)\|_{r'} \leq 2\gamma p \{ \| |u_j - v_j| (|u_j|^{2(p-1)} + |v_j|^{2(p-1)}) \|_{r'} + \tag{24}$$

$$\sum_{k=1, k \neq j}^m \| (|u_k - v_k| + |u_j - v_j|) (|u_j|^{2(p-1)} + |u_k|^{2(p-1)} + |v_j|^{2(p-1)} + |v_k|^{2(p-1)}) \|_{r'} \}.$$

Together Hölder and Minkowski inequalities imply

$$\begin{aligned} \| |u_j - v_j| (|u_j|^{2(p-1)} + |v_j|^{2(p-1)}) \|_{r'} &\leq \|u_j - v_j\|_r \| |u_j|^{2(p-1)} + |v_j|^{2(p-1)} \|_{\frac{r}{2(p-1)}} \\ &\leq \|u_j - v_j\|_r \{ \|u_j\|_r^{2(p-1)} + \|v_j\|_r^{2(p-1)} \}. \end{aligned}$$

In a similar way, we obtain

$$\| (|u_k - v_k| + |u_j - v_j|) \{ |u_j|^{2(p-1)} + |u_k|^{2(p-1)} + |v_j|^{2(p-1)} + |v_k|^{2(p-1)} \} \|_{r'} \leq$$

$$(\|u_k - v_k\|_r + \|u_j - v_j\|_r) \left\{ \sum_{h \in \{u_k, u_j, v_k, v_j\}} \|h\|_r^{2(p-1)} \right\}.$$

Combining with (24), we obtain

$$\|\nu_j(u) - \nu_j(v)\|_{r'} \leq 2\gamma p \sum_{k=1}^m \left\{ \sum_{f \in \{u_k, u_j, v_k, v_j\}} \|f\|_r^{2(p-1)} \right\} (\|u_k - v_k\|_r + \|u_j - v_j\|_r).$$

Since, we have

$$\|\nu_j(u) - \nu_j(v)\|_{\mathbf{N}([-T, T] \times \mathbb{R})} \leq \|\nu_j(u) - \nu_j(v)\|_{L^{q'} L^{r'}([-T, T] \times \mathbb{R})},$$

then

$$\|\nu_j(u) - \nu_j(v)\|_{\mathbf{N}([-T, T] \times \mathbb{R})} \leq 2\gamma p \sum_{k=1}^m \left\{ \sum_{f \in \{u_j, u_k, v_j, v_k\}} \|f\|_r^{2(p-1)} \right\} (\|u_j - v_j\|_r + \|u_k - v_k\|_r) \|_{q'}. \tag{25}$$

Recall that $q = \frac{4p}{p-1}$, since $q' = \frac{q}{q-1}$ then $\frac{1}{q'} > \frac{2p-1}{q}$. Consequently, it is possible to choose $\alpha > 0$ such that

$$\frac{1}{q'} = \frac{2p-1}{q} + \frac{1}{\alpha}.$$

Using the Hölder inequality, we get

$$\| \|u_j - v_j\|_r \|u_j\|_r^{2(p-1)} \|_{q'} \leq (2T)^{\frac{1}{\alpha}} \| \|u_j - v_j\|_r \|u_j\|_r^{2(p-1)} \|_{\frac{q}{2p-1}}.$$

With the Hölder inequality another time with respect to

$$\frac{2p-1}{q} = \frac{1}{q} + \frac{2(p-1)}{q},$$

we obtain

$$\| \|u_j - v_j\|_r \|u_j\|_r^{2(p-1)} \|_{q'} \leq (2T)^{\frac{1}{\alpha}} \|u_j - v_j\|_{L^q L^r([-T, T] \times \mathbb{R})} \|u_j\|_{L^q L^r([-T, T] \times \mathbb{R})}^{2(p-1)}. \quad (26)$$

From previous estimates (25) and analogous estimates as (26), derives that

$$\begin{aligned} \|\nu_j(u) - \nu_j(v)\|_{\mathbf{N}([-T, T] \times \mathbb{R})} &\leq 2\gamma p (2T)^{\frac{1}{\alpha}} \sum_{k=1}^m (\|u_j - v_j\|_{L^q L^r([-T, T] \times \mathbb{R})} \\ &\quad + \|u_k - v_k\|_{L^q L^r([-T, T] \times \mathbb{R})}) \left\{ \sum_{f \in \{u_j, u_k, v_j, v_k\}} \|f\|_{L^q L^r([-T, T] \times \mathbb{R})}^{2(p-1)} \right\}. \end{aligned}$$

Taking sup then sup, it follows that

$$\|\nu(u) - \nu(v)\|_{\mathbf{F}} \leq 8m\gamma p (2T)^{\frac{1}{\alpha}} \|u - v\|_{\mathbf{E}} \{ \|u\|_{\mathbf{E}}^{2(p-1)} + \|v\|_{\mathbf{E}}^{2(p-1)} \}.$$

Since $u, v \in B_{2C_1 R} \subset \mathbf{E}$, then

$$\|\nu(u) - \nu(v)\|_{\mathbf{F}} \leq 16m\gamma p (2T)^{\frac{1}{\alpha}} (2C_1 R)^{2(p-1)} \|u - v\|_{\mathbf{E}}.$$

Hence, by choosing T such that

$$16m\gamma p (2T)^{\frac{1}{\alpha}} (2C_1 R)^{2(p-1)} = \frac{1}{2C_1},$$

we get

$$\|\nu(u) - \nu(v)\|_{\mathbf{F}} \leq \frac{1}{2C_1} \|u - v\|_{\mathbf{E}}.$$

Therefore, assumptions of Theorem 4 are satisfied, then iteration method allows us to construct a unique local solution in $B_{2C_1 R}$ for the coupled problem $(CNLS)_p$. In addition, same arguments as in the proof of Theorem 1 give us uniqueness of local solution for our Cauchy problem with initial data in $L^2(\mathbb{R})^m$. Also, the continuity dependence from initial data will be produced from combining the inequality (9) and the first inequality from (23)

$$\|S(t)\psi_j\|_{\mathbf{S}(I \times \mathbb{R})} \leq C_1 \|\psi_j\|_2.$$

This complete the proof of the local well-posedness of the system $(CNLS)_p$ when initial data is in $L^2(\mathbb{R})^m$.

4.6. Proof of Lemma 4

At first we assume that $u \in C([-T, T], H^1(\mathbb{R})^m)$. Such solution for the problem $(CNLS)_p$ with initial data $\psi \in H^1(\mathbb{R})^m$ exists, one can see [12] for the defocusing problem and [13] for the focusing one. We multiply by u_j the convenient equation from the system $(CNLS)_p$, then conservation of the mass (3) follows by integration by parts. Hence, using density arguments give us the conservation result with initial data $\psi \in L^2(\mathbb{R})^m$.

4.7. Proof of Lemma 5

Let $\bar{T} < \infty$. Assume that for all j we have

$$\exists R^j > 0, \exists (t_n^j)_{n \in \mathbb{N}}, \lim_{n \rightarrow +\infty} t_n^j = \bar{T} \text{ and } \|u_j(t_n^j)\|_2 \leq R^j.$$

Taking $R = \sup_j R^j$. By some diagonal extraction techniques using the family of sequences $(t_n^j)_{n \in \mathbb{N}}$, one can find a sequence $(t_n)_{n \in \mathbb{N}}$ which does not depend on j and satisfies

$$\lim_{n \rightarrow +\infty} t_n = \bar{T} \text{ and } \|u_j(t_n)\|_2 \leq R.$$

We denote by $[0, T(R)[$ the maximal existence interval of nonnegative time for a solution with initial data $\psi \in L^2(\mathbb{R})^m$ such that $\|\psi\|_{L^2(\mathbb{R})^m} \leq R$. We take $k \in \mathbb{N}$ such that $t_k + T(R) > \bar{T}$. By Theorem 2 and starting from $\psi = u(t_k)$, we can extend the solution u up to $t_k + T(R)$. This contradicts the maximality of the existence interval $[0, T(R)[$. In the same way, we treat the assumption $\underline{T} < \infty$.

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